Physics Graduate School Qualifying Examination

Spring 2020 Part I

Instructions: Work all problems. This is a closed book examination. Calculators may not be used. **Start each problem on a new pack of correspondingly numbered paper and use only one side of each sheet.** **Place your 3-digit identification number in the upper right hand corner of each and every answer sheet. All sheets, which you receive, should be handed in, even if blank.** Your 3-digit ID number is located on your envelope. All problems carry the same weight.

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- Use correct vector notation when appropriate.
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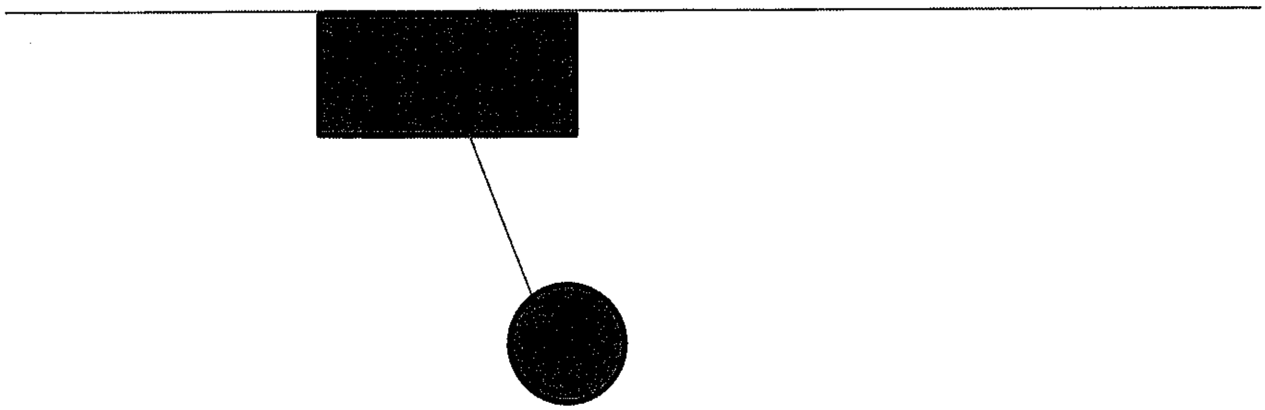
1. A spacecraft of mass m_1 is in a circular orbit of radius R about a planet with mass M_0; assume that $m_1 << M_0$. It applies its engines to supply an instantaneous impulse, and the speed is instantaneously increased by a factor of α (i.e., $v \rightarrow \alpha v$). For what values of α will the spacecraft escape to infinity?



2) Consider a cart of mass M_1 that hangs from a frictionless horizontal wire. There is a pendulum with mass m_2 that hangs from the cart on a wire of length L. A time-independent horizontal force, $F_x = -\partial V / \partial x$, is applied to the cart. The gravitational acceleration is g.

A) Write down the Lagrangian of the system using the horizontal position of the cart as x and the angle away from straight down for the pendulum as θ.

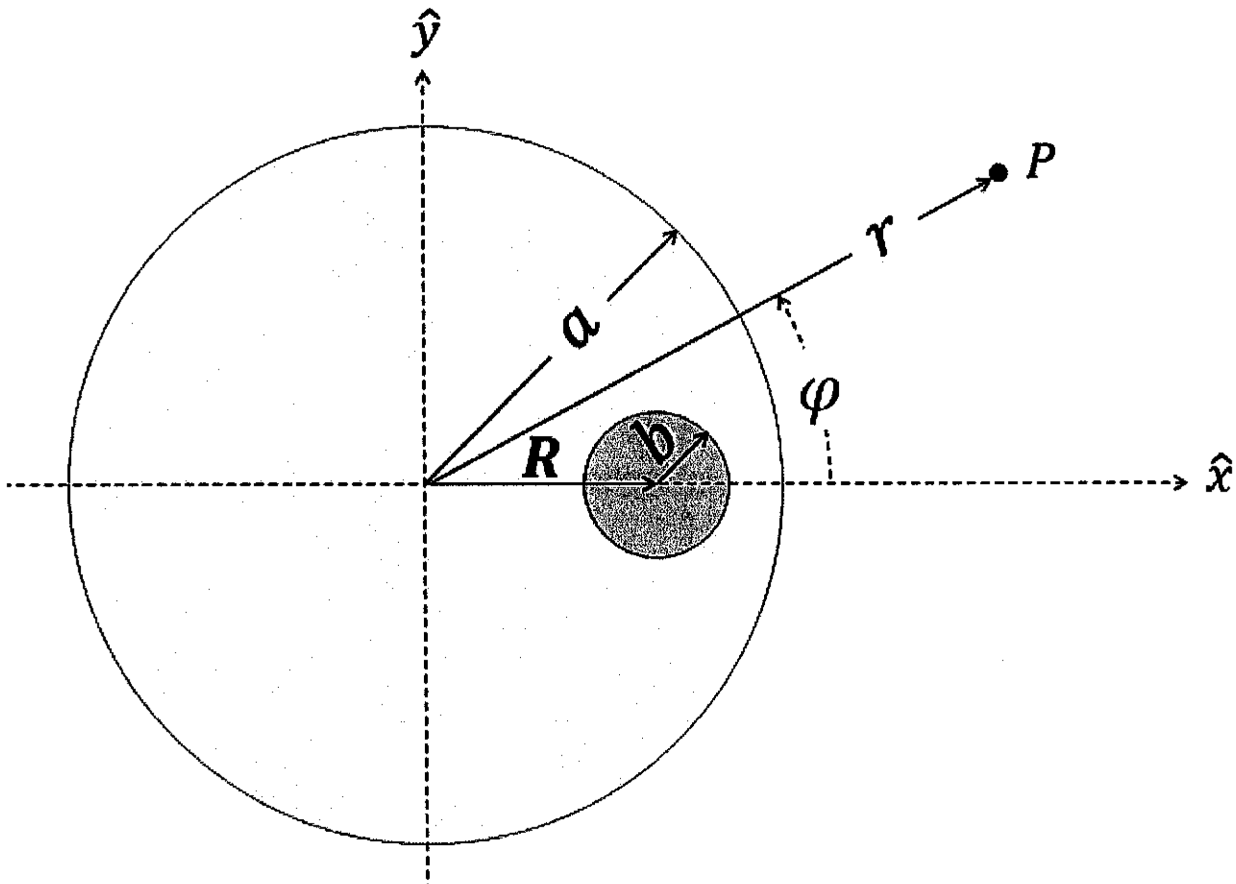
B) Write down the two equations of motion for the system.

C) Suppose we now set the horizontal force equal to zero. The equations of motion are coupled, but find a differential equation that depends only on θ.

3. A uniformly charged sphere (charge density ρ) of radius R is centered on the origin ($\vec{r} = 0$); it contains a spherical cavity of radius a, whose center is a distance d away from the center of the charged sphere (d+a<R). The center of the cavity is at $\vec{r} = d\hat{z}$.

a. Calculate the electric field inside the cavity.
b. Calculate the electric field for r<R, but outside the cavity.
c. Calculate the electric field for all points outside the charged sphere (r>R).

4) An infinitely long cable lies along the z-axis; it carries net current $\boldsymbol{I}$ (in the +z direction, out of the page) in the larger cross section of radius $\boldsymbol{a}$ and net current $\boldsymbol{I}$ (in the –z direction, into the page) in the smaller cross section of radius $\boldsymbol{b}$ $(< a - R)$. The smaller cross section is located a distance $\boldsymbol{R}$ $(< a)$ along the x-axis from the center of the larger cross section as shown in the figure.



Find the magnetic field $\vec{\boldsymbol{B}}(r, \varphi)$ at point P located at any $|\vec{r}| > a$ and $0 < \varphi < 2\pi$.

(5) A quantum particle of mass m is in the ground state of a one-dimensional harmonic oscillator potential, $V(x) = \frac{1}{2} m\omega^2 x^2$. At $t = 0$, the potential is suddenly changed to $V_1(x) = 2m\omega^2 x^2$.

(a) Compute the probability of this particle being in the ground state of $V_1(x)$.

(b) Compute the probability of this particle being in the first excited state of $V_1(x)$.

(c) At $t = T$, the potential is changed back to $V(x) = \frac{1}{2} m\omega^2 x^2$, and it is found that this particle returns to the ground state of $V(x)$ with 100 percent probability. Compute the smallest possible non-zero value of T.

6) At $t = 0$, an electron, with magnetic moment $\vec{\mu} = \frac{-e\hbar}{2mc}\,\vec{\sigma}$, is in the spin state

$$\chi(t=0) = \begin{pmatrix} i\sqrt{2/3} \\ \sqrt{1/3} \end{pmatrix}$$

A magnetic field B is applied in the z direction.

a) Find the spin state of the particle, as a function of time.

b) Find the expectation value of S_y as a function of time.

c) What is the probability, as a function of time, to measure that the electron's spin along the x direction is $\hbar/_2$?

7) Consider a classical, 3-dimensional gas with N particles; each particle has an energy given by $\varepsilon(\vec{p}) = ap^4$, where p is the magnitude of the particle's momentum. There are no interactions among the particles. Assume that the gas is held within a volume V and is at temperature T. The classical partition function takes the form:

$$Z(N,T,V) = c(N)V^{\psi}T^{\phi} .$$

a) Calculate ϕ and ψ .

b) Calculate the average energy, U(N,V,T).

c) Calculate the pressure, p(N,T,V).

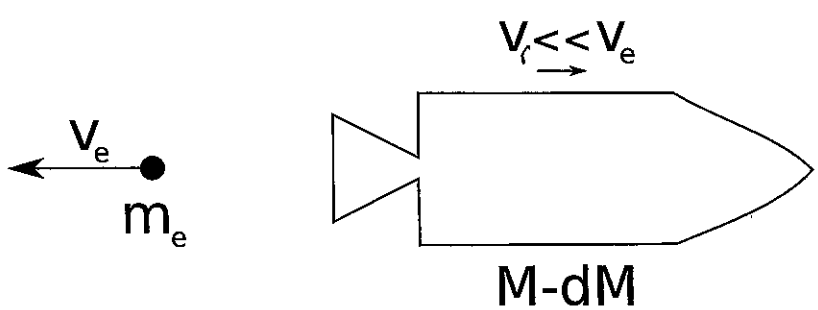
8) Relativistic Rocket: Consider a rocket of rest mass M that, in the rest frame of the rocket, exhausts particles of rest mass $m_e << M$ at a speed v_e (considered relativistic). We use V to denote the rocket's speed in the lab frame, and v_r to denote the rest frame speed.

a) In the rest frame, what is the velocity increment $dv_r << v_e$ that the rocket acquires when emitting just one particle? And, what is the change dM in the rest mass of the rocket? Compute dv_r/dM.

b) Using a), if the rocket is initially moving at speed V, what is the increase in velocity dV from emitting one particle, compute dV/dM.

c) If the rocket is initially at rest and it starts emitting particles until its rest mass is reduced to M/2, what is the final velocity V_f of the rocket?

d) What is V_f when v_e=c?



Physics Graduate School Qualifying Examination

Spring 2020 Part II

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1) Consider a particle of mass M, moving in two dimensions. Use polar coordinates r, θ. The particle moves in the presence of a central potential U(r), with $F(r) \equiv -dU/dr$.

a) Using the fact that $\ell = mr^2\dot{\theta}$ is a constant of the motion, derive an equation of motion for r(t) which does not involve θ.

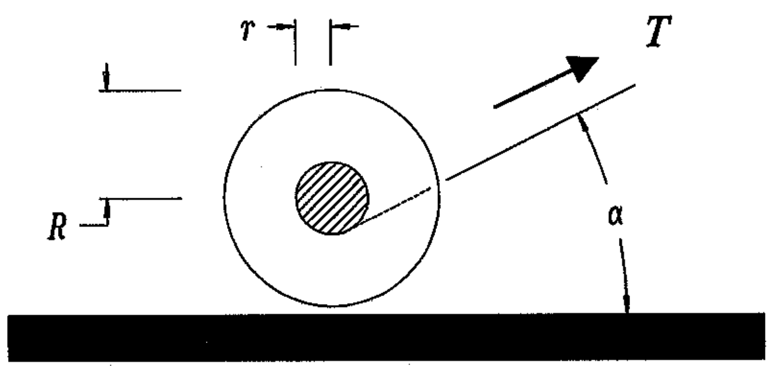
b) Suppose that the particle is in a circular orbit of radius r=b. Write ℓ as a function of b, without using θ.

c) Suppose the orbit is slightly perturbed, so that $r(t) = b + s(t)$, with s(t) very small. Working to first order in s(t), derive an equation of the form

$$\ddot{s} + \omega_0^2 s = 0$$

where ω_0 does not depend on s. Express the frequency ω_0 in terms of M, b, and the function F.

2) A spool of thread rests on a horizontal table. The end of the thread is pulled upward at an angle α to the horizontal. The thread passes under the core of the spool, and is pulled with tension T. The core radius is r and the outer radius is R. The spool's mass is m and its moment of inertia is I. The mass of the thread is negligible. At all times, the spool must not be lifted off the table, and rolls without slipping. The gravitational acceleration is given by g.

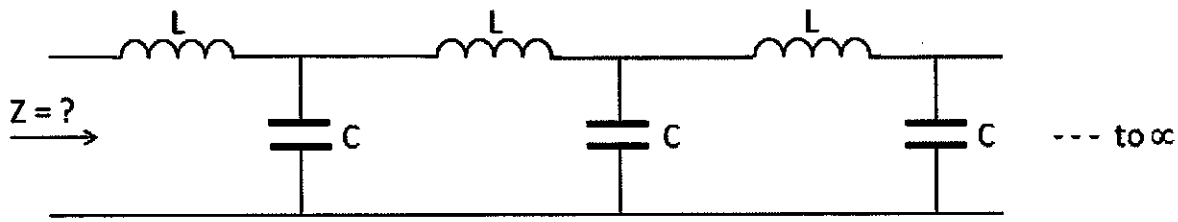


Set up a coordinate system where x points to the right on the above diagram.

(a) Find an expression for the friction force and its direction, in terms of the spool's parameters and the tension T.

(b) Calculate the linear acceleration $\ddot{x}$.

3) A long distance transmission line can be modeled as an infinite sequence of inductive and capacitive elements as shown.



What is the impedance, $\boldsymbol{Z}$ of the transmission line as a function of the angular frequency, ω of the voltage source, and the values of the inductor (L) and capacitors (C) as shown in the figure above?

4) A uniformly charged sphere (charge density ρ) of radius R rotates at a constant angular velocity ω, around the z-axis. The center of the sphere is at the origin, x=y=z=0.

The magnetic field along the z-axis, for z>R, can be written as an integral in spherical coordinates, as follows:

$$\vec{B}(z) = \hat{z}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{R} dr F(z,r,\phi,\theta)$$

What is the function $F(z,r,\phi,\theta)$?

(5) Consider the one-dimensional problem of a quantum particle of mass m moving in a delta-function potential, $U(x) = -V\,\delta(x)$, where $V > 0$ and $\delta(x)$ is the Dirac delta function.

(a) Write down the boundary conditions that the wavefunction must satisfy at $x = 0$.

(b) Compute the energy of the bound state.

6) A particle of mass M is in a one dimensional infinite square well; V(x)=0 for 0 < x < a, and V(x) is infinite for all other x. The wave function is given by $\psi(x) = Bx(a - x)e^{ikx}$. Here, k is real.

a) Compute B.

b) Compute the average energy we would measure for a particle in this state.

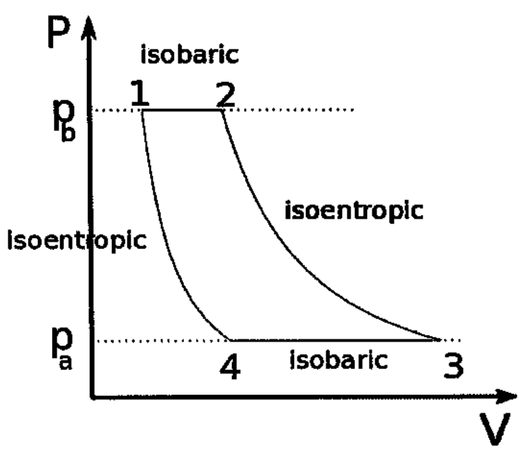
7) Consider a thermal engine where an ideal gas undergoes the particular cycle given in the PV diagram shown in the figure. It has two isobaric (constant pressure) and two isoentropic (constant entropy) paths. The (absolute) temperatures $T_{1,2,3,4}$ at the points 1,2,3,4 and the specific heat $C_P = C_V + Nk_B$ are the data of the problem.

a) For each leg of the cycle determine the heat Q absorbed. Take Q to be positive if the gas absorbs heat, and negative if it emits heat.

b) Compute the total work W produced by the engine.

c) Compute its efficiency (defined as $\eta = W/Q_a$), in terms of the four defined temperatures. Here Q_a is the total heat absorbed by the gas during the legs for which Q is positive.

c) Use the previous result to derive an inequality involving T_3-T_4 and T_2-T_1.



8) You are given three beakers filled with equal amounts of water at temperatures $T_1 > T_2 > T_3$, as well as access to a heat reservoir at a temperature, T_R. All temperatures lie within a range where water does not undergo any phase transition. The heat capacity of the water in each beaker is C, assumed to be temperature-independent. Assume that the heat capacities of the beakers are negligible.

We will consider a process in which all three beakers of water are put in contact with the reservoir, and brought to temperature T_R.

(a) Calculate the total entropy change of the water in the beakers when all three beakers are equilibrated with the reservoir.

(b) What value of T_R gives the smallest value for the entropy increase of the universe for this process?

(c) What is this minimum increase in entropy?