

Physics Graduate School Qualifying Examination

Fall 2020 Part I

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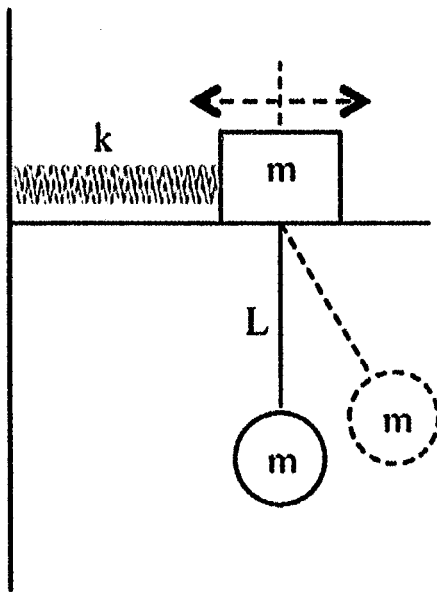
1) Consider a rocket of initial mass, m_0 , and initial velocity, $v_0 \hat{x}$, that burns fuel at a constant rate, $dm/dt = -b$ ($b > 0$), and expels it at a velocity, $-u \hat{x}$, relative to the rocket.

- Derive an equation that expresses the rocket's velocity as a function of time
- Now assume the rocket is in a uniform gravitational field with acceleration, $-g \hat{x}$.
Modify the equation in part A to express the rocket's velocity as a function of time

2) As shown in the figure, a pendulum of length L and mass m is connected to a block also of mass m that is free to move horizontally on a frictionless surface. The block is connected to a wall with a spring of spring constant k . The gravitational acceleration is given by g . For the special case where $k/m=g/L=\omega_0^2$, determine:

(a) The frequencies of the normal modes of this system for small oscillations around the equilibrium positions.

(b) If A is the amplitude of the oscillation for the angle of the pendulum, and B is the amplitude for the displacement oscillation of the block, determine, for each normal mode, the ratio of the two amplitudes A/B .



3) A plane electromagnetic wave of angular frequency ω propagates in a uniform plasma with electron density n_e and local charge neutrality: $\rho = 0$ everywhere. The EM wave generates periodic currents within the plasma, so that the index of refraction of the medium differs from that of the vacuum.

Assume the plasma has no resistivity. Moreover, assume the ions are much heavier than the electrons so that we can neglect the current due to the ions.

Thus, we take $\vec{E}(\vec{r}, t) = E_0 \hat{x} e^{i(kz - \omega t)}$; $\vec{B}(\vec{r}, t) = \vec{B}_0 e^{i(kz - \omega t)}$ and

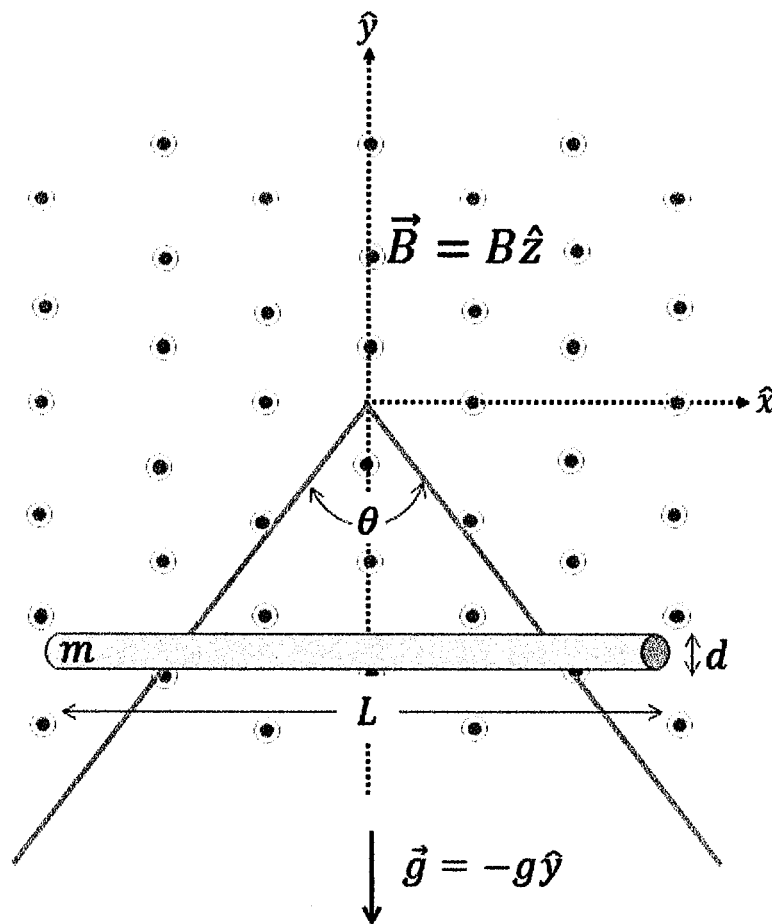
$$\vec{J}(\vec{r}, t) = \vec{J}_0 e^{i(kz - \omega t)}$$

a) Calculate the response of the electrons (using $\vec{F} = m\vec{a}$), and so compute the vector \vec{J}_0 in terms of E_0 , ω , k , and fundamental constants. Ignore the magnetic force on the electrons.

b) Using Maxwell's equations, compute the vector \vec{B}_0 in terms of E_0 , ω , k , and fundamental constants.

c) Find the frequency ω as a function of k .

4) A rigid rod of mass m , resistivity ρ , length L , diameter d is released from rest from the vertex of a V-shaped rail of negligible electrical resistance. The rail has a vertex angle θ at the origin as shown in the figure. The system is in the presence of a constant magnetic field $\vec{B} = B\hat{z}$ and acceleration due to gravity is $\vec{g} = -g\hat{y}$. The rod is released at time $t = 0$. Neglecting the effects of friction or air resistance, find the differential equation for the position of the center of mass of the rod, $\vec{r}(t)$ as a function of time until the rod loses contact with the rails. Note: You do not need to solve the differential equation for $\vec{r}(t)$, you just need to set it up.



(5) Using the orthogonal basis states, $|1\rangle, |2\rangle, |3\rangle$, all of which are normalized to 1, the matrix representation of the Hamiltonian of a three-level system is written as

$$H_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta \end{pmatrix},$$

where $\Delta > 0$. Now an additional term,

$$H' = \begin{pmatrix} 0 & 0 & \Omega_1 \\ 0 & 0 & \Omega_2 \\ \Omega_1 & \Omega_2 & 0 \end{pmatrix},$$

is added such that the new Hamiltonian is given by $H = H_0 + H'$. Ω_1 and Ω_2 are real and finite.

(a) One eigenstate of H , which is denoted as $|1'\rangle$, always has zero energy for any $\Omega_1, \Omega_2, \Delta$. Write down the expression of $|1'\rangle$ in terms of $|1\rangle, |2\rangle, |3\rangle$.

(b) A state, $|2'\rangle$, is normalized to 1, is a linear combination of $|1\rangle$ and $|2\rangle$, and is orthogonal to $|1'\rangle$. Write down the expression of $|2'\rangle$ in terms of $|1\rangle, |2\rangle$.

(c) Using $|1'\rangle, |2'\rangle, |3\rangle$, rewrite the matrix representation of H .

(d) Using the result obtained in (c), evaluate the other two eigenenergies of H .

6) A particle of mass m is constrained, in three dimensions, between two concentric, non-penetrable spheres of radius a and b , with $a < b$.

Find the ground state wavefunction, and the ground-state energy.

7) Consider the following system. It consists of N **distinguishable** objects. Each object can be in two states, one state with energy zero, the other state with energy \mathcal{E} .

a) Compute the partition function $Z(N, T)$ for this system, and then the free energy $F(N, T)$.

b) Compute the entropy $S(N, T)$; compute its value as T approaches infinity.

c) Compute the energy $U(N, T)$; compute its value as T approaches infinity.

8) Consider a 'molecular zipper' with N links, maintained at temperature T . Each link has two states, 'open' with energy ϵ , and 'closed' with zero energy. In addition, a link cannot close unless all links to its left are also closed.

a) Find the partition function of the zipper.

b) What is the probability that the second link on the *right* end is open, when $N \rightarrow \infty$?

c) What is the entropy of the zipper, when $N \rightarrow \infty$?

Useful Equations

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \quad H = -\vec{\mu} \cdot \vec{B}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3 r' \frac{\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \quad \oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc}$$

$$\mathcal{E} = -d\Phi / dt$$

$$dU = TdS - pdV + \mu dN \quad dF = -SdT - pdV + \mu dN$$

$$F = U - TS = -kT \ln Z \quad Z = \sum_i e^{-\frac{E_i}{kT}} \quad p_i = e^{-\frac{E_i}{kT}} / Z$$

$$Z_{classical} = \frac{1}{h^{3N} N!} \int d^{3N} p d^{3N} r e^{-H/kT} \quad \Delta U = C_V \Delta T$$

$$L = K - V \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi \quad -\frac{\hbar^2}{2M} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi = E\psi$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In spherical coordinates:

$$\nabla^2 f = \frac{1}{r} \frac{\partial^2}{\partial r^2} (rf) + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} (\sin(\theta) \frac{\partial f}{\partial \theta}) + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2 f}{\partial \phi^2}$$

Harmonic Oscillator: $\psi_0(x) = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}$

$$\int_{-\infty}^{\infty} dx e^{-x^2} = \sqrt{\pi}$$

$$\int_0^{\infty} dx x^n e^{-x} = n!$$

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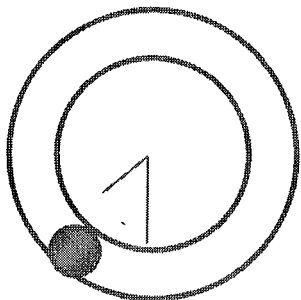
Fall 2020 Part II

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1) Consider a bead inside a torus that is hollow. The bead slides smoothly with no friction in the torus. The torus has mass, M_1 , and radius, R , and rolls without slipping with angular speed, $\dot{\phi}$. The bead has mass, M_2 , and moves with angular speed, relative to the torus, of $\dot{\theta}$. The size of the bead is much smaller than R , and the mass of the torus is concentrated at radius R . Note that we define the angle of the bead, relative to a fixed position on the torus, as θ .

- Write down the total kinetic energy of the torus (translation and rotational).
- Find the kinetic energy of the bead.
- Write down the potential energy of the system in a gravitational field with acceleration g .
- Using the Lagrangian you can construct from the answers in A, B, and C, find the two equation of motion equations for the coordinates.



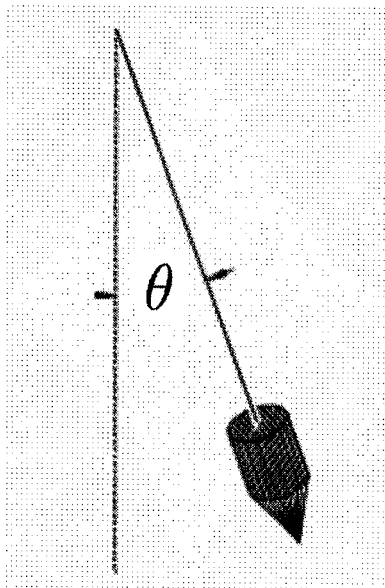
2. A suspended bob of mass M hangs from a string. The length of the string attached to the bob is L . The bob is situated in a train car moving on a level surface, but accelerating at $-g / \sqrt{3}$

along the track. Thus we take the gravitational acceleration to be $-g\hat{z}$, and the acceleration of the train car to be $-g(3)^{-1/2}\hat{x}$

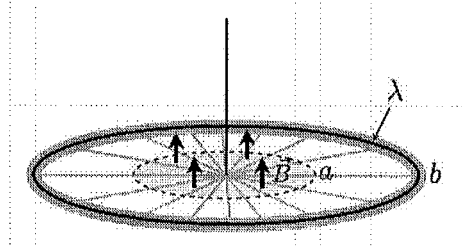
where z is in the vertical direction and x is in the horizontal direction.

a) In static equilibrium, the bob will deviate from the vertical, at an equilibrium angle θ_0 . Compute this angle.

b) Now imagine you set the bob swinging about this equilibrium angle. Calculate the period for the bob's oscillation, assuming that $\theta - \theta_0$ is small.



3) The rim of a wheel of radius b is charged with a linear charge density λ . The wheel is suspended horizontally and is free to rotate with no friction. The spokes and the wheel are made of non-conducting materials. In the central region out to a radius $a < b$ is a uniform magnetic field B pointing up; see Figure below.



a) Explain qualitatively what happens to the wheel when somebody turns the B -field off.

b) Compute the resulting angular momentum given to the wheel.

(Ignore any effects of the magnetic field generated by the charge density on the wheel.)

4) A rigid disk of radius R , thickness d , and charge density

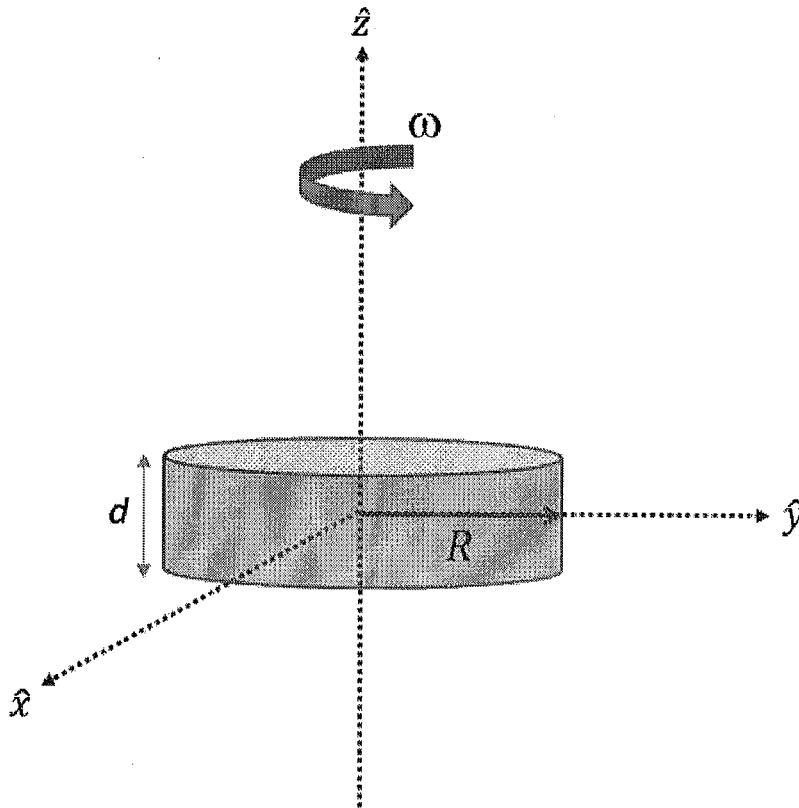
$\rho(r) = \rho_0 (1 - r / R)$ is located with its center at the origin. The disk is rotating as shown with constant angular velocity $\omega \hat{z}$; this rotation creates a magnetic field. We may write $B_z(0,0,a)$, the z -component of the magnetic field at the point $x=y=0, z=a$, in the following form (assume $a > d/2$):

$$B_z(0,0,a) = \int_0^{2\pi} d\phi' \int_{-d/2}^{d/2} dz' \int_0^R dr' F(r', z', \phi', a)$$

where we use cylindrical coordinates:

$$x = r \cos(\phi), y = r \sin(\phi), z = z$$

Find the function $F(r, z, \phi, a)$



(5) The Hamiltonian of two identical spin-1/2 particles is written as $H = J \vec{s}_1 \cdot \vec{s}_2 = J(s_{1x}s_{2x} + s_{1y}s_{2y} + s_{1z}s_{2z})$, where J is a

constant, $s_{ix} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $s_{iy} = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $s_{iz} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $i = 1, 2$. The two eigenstates of s_{iz} with eigenvalues of $1/2$ and $-1/2$ are denoted by $|\uparrow\rangle_i$ and $|\downarrow\rangle_i$, respectively. $\hbar = 1$ has been chosen to simplify the notations.

(a) Compute the eigenenergies and eigenstates of H . (Hint:

consider the operator $S^2 = (\vec{s}_1 + \vec{s}_2) \cdot (\vec{s}_1 + \vec{s}_2)$)

(b) If the initial state of these two particles is

$|\psi(0)\rangle = \frac{1}{\sqrt{2}} (|\leftarrow\rangle_1 |\rightarrow\rangle_2 - |\rightarrow\rangle_1 |\leftarrow\rangle_2)$, compute the wavefunction at time t , where $|\leftarrow\rangle_i$ and $|\rightarrow\rangle_i$ are the eigenstates of s_{ix} with eigenvalues of $1/2$ and $-1/2$, respectively.

(c) Evaluate the expectation value of $s_{1x} + s_{2x}$ at time t .

6. The Hamiltonian of a spinless particle, with charge Q and mass m , in a magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ is given by $H = \frac{1}{2m} (\vec{p} - Q\vec{A}(\vec{r}))^2$, where $\vec{p} = (p_x, p_y, p_z)$ is the conjugate momentum of the particle position \vec{r} . Take $\vec{A} = -B_0 y \hat{x}$, corresponding to a uniform $\vec{B} = B_0 \hat{z}$.

- a. Assume that the wavefunction of an energy eigenstate has the following form: $\Psi(x, y, z) = e^{ik_x x + ik_z z} \phi(y)$. Derive the differential equation satisfied by $\phi(y)$.
- b. Using the equation derived in part (a), derive the allowed energy eigenvalues. (Hint: note the similarity of the equation satisfied by $\phi(y)$ to that which arises in the harmonic oscillator problem.)

7) Consider a monoatomic ideal gas with N atoms in a volume V at temperature T .

a) Compute the (classical) partition function Z .

b) Compute the mean value of the energy of a single atom $\langle E \rangle$.

c) Compute the mean value of the energy fluctuation

$$\sqrt{\langle \Delta E^2 \rangle} = (\langle E^2 \rangle - \langle E \rangle^2)^{\frac{1}{2}}$$

d) Consider a given volume $v \ll V$. Compute the probability of finding the volume v empty (with no molecules).

8) Suppose we have three solid bodies that are isolated from the rest of the universe; initially, they are not in contact with each other, and are separately in equilibrium at temperatures T_1 , T_2 , and T_3 .

The constant volume heat capacities are C_1 , C_2 , and C_3 , and they are independent of temperature.

We now place them in thermal contact with each other, and let them reach a new equilibrium at a final temperature T_F . Assume that the volumes of the bodies do not change.

a) Compute the final temperature T_F .

b) Compute the total change in entropy, between the initial and final configurations.

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